Appendix B

Proof of a Geometric Relationship

The purpose of this Appendix is to prove a differential relationship central to the derivation of the integral kernel for incident angular fluxes. We begin with the local coordinate system illustrated in Fig. 1. We define $s_b(\overrightarrow{\Omega})$ to be the distance to the outer boundary from the point $\overrightarrow{r}$ along the direction, $-\overrightarrow{\Omega}$. For each direction, $\overrightarrow{\Omega}$, the vector $\overrightarrow{r_b}$, where

$$\overrightarrow{r}_b = \overrightarrow{r} - s_b \overrightarrow{\Omega} \,. \tag{1}$$

represents a point on the outer surface of the transport domain. Assuming a non-reentrant transport domain, each point on the outer surface corresponds to a unique value of $\overrightarrow{\Omega}$, and each value of $\overrightarrow{\Omega}$ corresponds to a unique point on the outer surface. It follows from basic principles of calculus that

$$dA = \left\| \frac{\partial \overrightarrow{r}_b}{\partial \mu} \times \frac{\partial \overrightarrow{r}_b}{\partial \Phi} \right\| \, d\mu \, d\Phi \,, \tag{2}$$

where dA is the differential surface area associated with the surface point $\overrightarrow{r}_b$, $\overrightarrow{\Omega}$ is defined by the polar and azimuthal angles, θ and Φ, respectively, and $d\mu\, d\Phi = d\Omega$, where $\mu = \cos\theta$. The outward-directed surface normal at point $\overrightarrow{r}_b$ is given by

$$\overrightarrow{n} = \left(\frac{\partial \overrightarrow{r}_b}{\partial \mu} \times \frac{\partial \overrightarrow{r}_b}{\partial \Phi} \right) \Big/ \left\| \frac{\partial \overrightarrow{r}_b}{\partial \mu} \times \frac{\partial \overrightarrow{r}_b}{\partial \Phi} \right\| \,. \tag{3}$$

Thus, using Eqs. (2) and (3), we find that

$$\left| \overrightarrow{\Omega} \cdot \overrightarrow{n} \right| \, dA = \left| \overrightarrow{\Omega} \cdot \left(\frac{\partial \overrightarrow{r}_b}{\partial \mu} \times \frac{\partial \overrightarrow{r}_b}{\partial \Phi} \right) \right| \, d\mu \, d\Phi \, . \tag{4}$$

From Eq. (1), it follows that

$$\frac{\partial \overrightarrow{r}_b}{\partial \mu} = - \left(\frac{\partial s_b}{\partial \mu} \overrightarrow{\Omega} + s_b \frac{\partial \overrightarrow{\Omega}}{\partial \mu} \right) \, , \tag{5a}$$

and that

$$\frac{\partial \overrightarrow{r}_b}{\partial \mu} = - \left(\frac{\partial s_b}{\partial \Phi} \overrightarrow{\Omega} + s_b \frac{\partial \overrightarrow{\Omega}}{\partial \Phi} \right) \, . \tag{5b}$$

Thus,

$$\begin{aligned} \frac{\partial \overrightarrow{r}_b}{\partial \mu} \times \frac{\partial \overrightarrow{r}_b}{\partial \Phi} \quad &= \quad \frac{\partial s_b}{\partial \mu} \frac{\partial s_b}{\partial \Phi} \left(\overrightarrow{\Omega} \times \overrightarrow{\Omega} \right) + \frac{\partial s_b}{\partial \mu} s_b \left(\overrightarrow{\Omega} \times \frac{\partial \overrightarrow{\Omega}}{\partial \Phi} \right) + \\ & \quad\; s_b \frac{\partial s_b}{\partial \Phi} \left(\frac{\partial \overrightarrow{\Omega}}{\partial \mu} \times \overrightarrow{\Omega} \right) + s_b^2 \left(\frac{\partial \overrightarrow{\Omega}}{\partial \mu} \times \frac{\partial \overrightarrow{\Omega}}{\partial \Phi} \right) \, . \end{aligned} \tag{6}$$

The first term on the right side of Eq. (6) is identically zero, and the second and third terms are orthogonal to $\overrightarrow{\Omega}$. Thus

$$\overrightarrow{\Omega} \cdot \left(\frac{\partial \overrightarrow{r}_b}{\partial \mu} \times \frac{\partial \overrightarrow{r}_b}{\partial \Phi} \right) = s_b^2 \overrightarrow{\Omega} \cdot \left(\frac{\partial \overrightarrow{\Omega}}{\partial \mu} \times \frac{\partial \overrightarrow{\Omega}}{\partial \Phi} \right) \, . \tag{7}$$

One finds by direct evaluation that

$$\left(\frac{\partial \overrightarrow{\Omega}}{\partial \mu} \times \frac{\partial \overrightarrow{\Omega}}{\partial \Phi} \right) = - \overrightarrow{\Omega} \, . \tag{8}$$

Substituting from Eq. (8) into Eq. (7), we find that

$$\overrightarrow{\Omega} \cdot \left(\frac{\partial \overrightarrow{r}_b}{\partial \mu} \times \frac{\partial \overrightarrow{r}_b}{\partial \Phi} \right) = - s_b^2 \, . \tag{9}$$

Finally, substituting from Eq. (9) into Eq. (4), we obtain the desired geometric relationship:

$$
\begin{aligned}
\left| \overrightarrow{\Omega} \cdot \overrightarrow{n} \right| dA &= s_b^2 \, d\mu \, d\Phi \, , \\
&= s_b^2 \, d\Omega \, .
\end{aligned} \tag{10}
$$

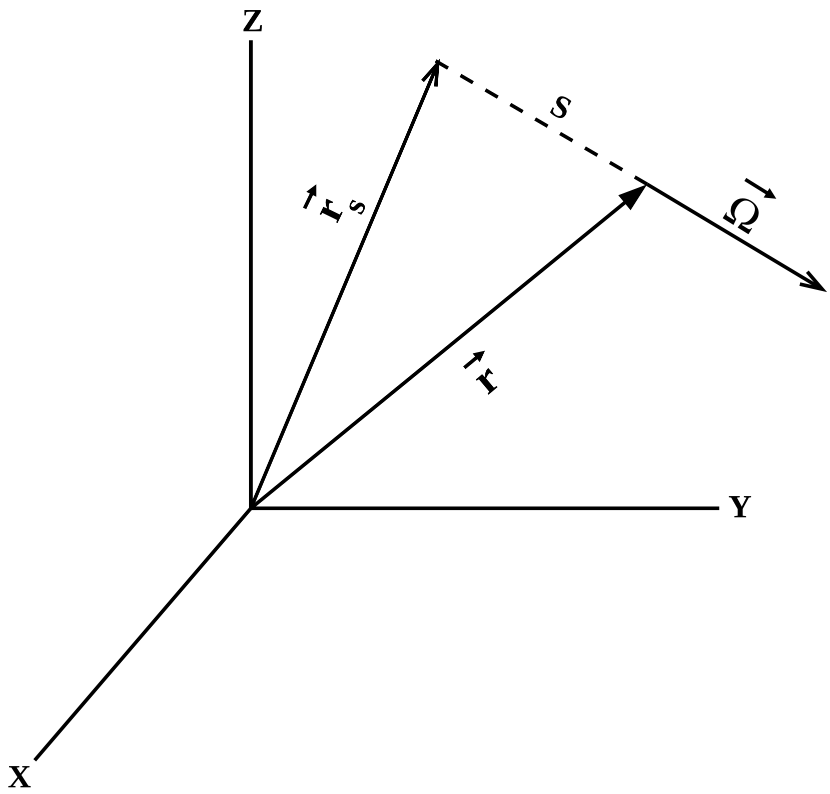

Figure 1: The local spatial coordinate system at the point $\overrightarrow{r}$. Note that $\overrightarrow{r}_s = \overrightarrow{r} - s\,\overrightarrow{\Omega}$. This is a form of local spherical coordinate system with $\overrightarrow{\Omega}$ playing the role of both a spatial and angular variable.